Two-level Fractional and Fractional Factorial Designs in Blocks of Size Two

Janet Godolphin

Department of Mathematics
University of Surrey

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Description

Consider a 2^p factorial treatment structure where practical constraints make it necessary to arrange the runs in blocks of size two.

- a batch of raw material may only be sufficient for two runs
- experiments using two-colour microarrays

A construction approach is given to generate designs, comprising full factorial replicates in blocks of size two, for estimation of main effects and two factor interactions.

The work is extended to include replicates of 2^{p-r} fractions.

Definition

The 2^p treatment combinations can be applied to 2^{p-1} blocks of size two so that

- 2^{p-1} factorial effects are orthogonal to blocks, and are estimable
- $2^{p-1} - 1$ factorial effects are completely confounded with block effects, and are inestimable.

Such an arrangement will be described as a blocked replicate.

It is assumed that blocks do not interact with factors.

Illustration: A blocked replicate of 2^4

(1)	a	b	ab	d	ad	bd	cd
abc	bc	ac	c	$abcd$	bcd	acd	abd

Effects confounded with blocks: AB, AC, BC, D, ABD, ACD, BCD.

Estimable effects: A, B, C, ABC, AD, BD, CD, $ABCD$.

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Replicate is generated by abc: the Selected Treatment Combination (STC)
Kerr (2006) - uses the term generator

Estimable effects have an odd number of terms in common with the STC.

Estimable effects of interest: A, B, C, AD, BD, CD.

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No single blocked replicate exists.

However, there are many pairs of blocked replicates which, with replicate with STC abc, enable estimation of all effects of interest.

Eg: cd and b:

C, D, AC, AD, BC, BD and B, AB, BC, BD.

In general.....

The smallest number of blocked replicates required to estimate all effects of interest is: $M = \lfloor \log_2 p \rfloor + 1$.

Aim: To produce a flexible design approach that gives designs in M blocked replicates.

Method - Stage 1

Stage 1 Choose $f_1, f_2, \ldots, f_M$ to satisfy:

$f_i \leq 2^{M-i}$ and $\sum_{i=1}^{M} f_i = p$.

Allocate f_i factors to set $\mathcal{F}_i$.

Example: Consider a 2^{12} experiment. This has $M = \lfloor \log_2 12 \rfloor + 1 = 4$. Select $f_1 = 6$, $f_2 = 3, f_2 = 2$ and $f_4 = 1$, and allocate factors as

$$\mathcal{F}_1 = \{A, B, C, D, E, F\}, \mathcal{F}_2 = \{G, H, I\}, \mathcal{F}_3 = \{J, K\}, \mathcal{F}_4 = \{L\}.$$

Method - Stage 2

Stage 2 Use the 'STC Inclusion Process' on each of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$.

Example: Use of STC Inclusion Process on $\mathcal{F}_1 = \{A, B, C, D, E, F\}$:
Allocate members of $\mathcal{F}_1$ to $2^{\lfloor log_2 f_1 \rfloor} = 4$ sets of sizes one and two. Eg:

$$S_1 = \{A, B\},\ S_2 = \{C\},\ S_3 = \{D\},\ S_4 = \{E, F\}.$$

$\boldsymbol{STC_1}$**:** Include all factors of $\mathcal{F}_1$.
$\boldsymbol{STC_2}$**:** Include factors of S_1 and S_2.
$\boldsymbol{STC_3}$**:** Include factors of S_1 and S_3.
$\boldsymbol{STC_4}$**:** Include first factor from each of S_1 and S_4 and any chosen factors from S_2 and S_3.

Method - Stage 2

STC Inclusion Process on $\mathcal{F}_1 = \{A, B, C, D, E, F\}$ gives:

	Factors of $\mathcal{F}_1$ to be included
STC_1	A, B, C, D, E, F
STC_2	A, B, C
STC_3	A, B, D
STC_4	$A, (C), (D), E$

Use STC Inclusion Process on $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$.

Method - Stage 3

Combine the results of Stage 2 to obtain STC for each blocked replicate.

	STC Inclusion				Selected
	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	*STC*
STC_1	A, B, C, D, E, F	-	-	-	*abcdef*
STC_2	A, B, C	G, H, I	-	-	*abcghi*
STC_3	A, B, D	G	J, K	-	*abdgjk*
STC_4	$A, (C), (D), E$	$(G), H$	J	L	*aeghjl*

Design Properties

Estimators for Main Effects				
No. of blocked replicates providing an estimator	1	2	3	4
Number of main effects	4	5	2	1
Estimators for Two Factor Interactions				
No. of blocked replicates providing an estimator	1	2	3	4
Number of two factor interactions	16	26	20	4

Results of Design Construction Process, $p = 4$

STC sets	A	B	C	D	AB	AC	AD	BC	BD	CD	n_m, n_t
abcd, *ab*, *ac*	3	2	2	1	1	1	2	2	1	1	**8, 8**
abc, *abd*, *a*	3	2	1	1	1	2	2	1	1	2	**7, 9**
abc, *abd*, *ad*	3	2	1	2	1	2	1	1	2	3	**8, 10**
abc, *abd*, *acd*	3	2	2	2	1	1	1	2	2	2	**9, 9**
abc, *cd*, *a*	2	1	2	1	1	2	3	1	2	1	**6, 10**
abc, *cd*, *ac*	2	1	3	1	1	1	3	2	2	2	**7, 11**
abc, *cd*, *ad*	2	1	2	2	1	2	2	1	3	2	**7, 11**
ab, *ac*, *d*	2	1	1	1	1	1	3	2	2	2	**5, 11**
ab, *ac*, *ad*	3	1	1	1	2	2	2	2	2	2	**6, 12**
ab, *ac*, *bd*	2	2	1	1	2	1	3	3	1	2	**6, 12**

See Yang and Draper (2003)

$p = 8$

	$D1$	$D2$ Kerr
STC_1	*abcd*	*abcdefgh*
STC_2	*abefg*	*abcd*
STC_3	*acefh*	*abef*
STC_4	*bcdegh*	*aceg*

Estimators for Main Effects					
No. of blocked replicates providing an estimator	1	2	3	4	n_m
Number of main effects $D1$	0	4	4	0	**20**
Number of main effects $D2$	1	3	3	1	**20**
Estimators for Two Factor Interactions					
No. of blocked replicates providing an estimator	1	2	3	4	n_t
Number of two factor interactions $D1$	8	11	8	1	**58**
Number of two factor interactions $D2$	12	12	4	0	**48**

Fractional Factorials

The method can be extended to provide a design construction approach that uses blocked fractional factorials.

First, identify M STCs for a design in blocked full replicates.

The fraction for a blocked replicate must be such that:

Condition 1 Every design generator has an even number of factors in common with the STC.
Condition 2 No effects of interest estimable from the blocked replicate are aliased with each other.

Fractional Factorials

Generators for 2^{12-5} Blocked Fractional Replicates.

STC	design generators
$abcdef$	$ABCE, ABDF, GHJ,$ GIK, HIL
$abcghi$	$ABCH, ABGI, DEJ$ DFK, EFL
$abdgjk$	$ABDJ, ABGK, CEH,$ CFI, EFL
$aeghjl$	$AEGJ, AEHL, BCF,$ BDI, CDK

Summary

- For blocked replicates, the process gives an alternative to existing methods and an improvement for some p.
- Not automatic to implement - practitioners can adapt the process to prioritise estimation of particular main effects or two factor interactions.
- Method extends to blocked fractional factorials.

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